MATH 2050C Lecture 16 (Mar 11)

[Reminder: Midterm starts today at 6:00PM.]

Recall: "ε-δ definition for limit of functions" $f: A \to \mathbb{R}$

$$\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 \text{ s.t.}$$

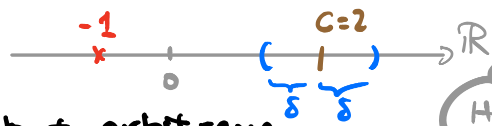
$$|f(x) - L| < \varepsilon \text{ whenever } x \in A \text{ and } 0 < |x - c| < \delta$$

cluster point

Example: $\boxed{\lim_{x \to 2} \frac{x^3 - 4}{x+1} = \frac{4}{3}}$

$f: A := \mathbb{R} \setminus \{-1\} \to \mathbb{R}$

$f(x) := \dfrac{x^3-4}{x+1}$



Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Note: If $|x - 2| < 1$, then

$$1 < x < 3$$

Hence, $|x+1| > 2 > 0$

and $|3x^2 + 6x + 8| \leq 1000$.

Choose $\delta = \min\{1, \frac{3}{500} \varepsilon\}$.

THEN, $\forall x \in A$, and $0 < |x - 2| < \delta$,

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right| = \left| \frac{3x^3 - 4x - 16}{3(x+1)} \right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot |x-2|$$

$$< \frac{1000}{3 \cdot 2} \delta \leq \varepsilon$$

$\blacksquare$

Here: $0 < |x-2| < \delta$

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right| = \left| \frac{3(x^3-4) - 4(x+1)}{3(x+1)} \right|$$

$$= \left| \frac{3x^3 - 4x - 16}{3(x+1)} \right|$$

$$= \left| \frac{(x-2)(3x^2+6x+8)}{3(x+1)} \right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot \underbrace{|x-2|}_{\text{small}}$$

Note: $|x-2| < 1$

$$\Rightarrow \quad 1 < |x| < 3$$

So $|x+1| > 2 > 0$.

and $|3x^2 + 6x + 8|$

$\leq 3|x|^2 + 6|x| + 8$

$\leq 3 \cdot 3^2 + 6 \cdot 3 + 8 \leq 1000$

Prop: $\lim_{x \to c} f(x)$, if exists, is unique. (Pf: Exercise!)

Thm: "Sequential Criteria"

$$\lim_{x \to c} f(x) = L \iff \forall \text{ seq. } (x_n) \text{ in } A \text{ s.t. } \overset{(*)}{\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}}$$

we have $\lim (f(x_n)) = L$

Proof: "$\Rightarrow$" Let (x_n) be a seq in A s.t. $(*)$ holds

Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim_{x \to c} f(x) = L$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad \begin{matrix} x \in A \\ 0 < |x - c| < \delta \end{matrix}$$

Since$^{(*)}$ $\lim (x_n) = c$, for the $\delta > 0$ above.

$\exists\, K = K(\delta) \in \mathbb{N}$ s.t. $0 \overset{(*)}{<} |x_n - c| < \delta \quad \forall n \geq K$

$\Rightarrow \quad |f(x_n) - L| < \varepsilon \quad \forall n \geq K$

"$\Leftarrow$" Suppose NOT, i.e. $\exists\, \varepsilon_0 > 0$ s.t. $\forall \delta > 0$,

$\exists\, x_\delta \in A$ s.t. $0 < |x_\delta - c| < \delta$

BUT: $\quad |f(x_\delta) - L| \geq \varepsilon_0$

Take $\delta = \frac{1}{n}$, then get $x_n \in A$ s.t.

$$0 < |x_n - c| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$$

$\Rightarrow \quad \lim (x_n) = c \qquad$ BUT $\quad \lim (f(x_n)) \neq L$

$x_n \neq c \quad \forall n \in \mathbb{N}$

Contradiction!

$\blacksquare$